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Irregular frequencies and iterative methods in the solution of steady surface-wave problems in hydrodynamics

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Summary

Two linearized problems in free-surface hydrodynamics are discussed. The first concerns flow due to a submerged line vortex in a running stream, and the second investigates three-dimensional flow about a moving pressure distribution at the surface of the fluid. Closed-form solutions to both linearized problems are well known, and therefore are not of interest; however, it is shown that the solution of either problem by a boundary-integral technique utilizing "simple" (Rankine) sources as fundamental singular solutions leads to Fredholm integral equations of the second kind, for which the irregular frequencies do not occur discretely, but as a *continuum*. Consequently, Neumann-type iteration schemes for the solution of these equations necessarily diverge for any Froude number. Ramifications of this result in the attempted numerical solution of the corresponding *non-linear* problems are discussed, and the convergence difficulties encountered by Hess [1] are analyzed.

1. Introduction

This study is primarily concerned with the numerical solution of the singular, Fredholm integral equations of the second kind which arise in the use of boundary-integral methods and surface-singularity techniques to formulate problems in potential flow. The use of such techniques is natural, since they possess the obvious advantage of reducing by one the number of dimensions involved in the statement of these problems. In addition, they permit the boundary condition either at infinite depth within the fluid or on some horizontal bottom to be satisfied identically.

Integral-equation methods have been used extensively to solve problems in hydrodynamics, and are reviewed by Yeung [2], with emphasis on applications involving free-surface flows. Some examples of two-dimensional, non-linear, free-surface flow problems which have yielded to numerical solution by these methods may be found in papers by Hess [1], Vanden Broeck and Tuck [3], Forbes and Schwartz [4], Forbes [5] and Smith and Abd-el-Malek [6]. Solutions to linearized problems in three dimensions have also been sought by integral-equation techniques, following the pioneering work of Hess and Smith [7,8]. Their method has been improved somewhat by Landweber and Macagno [9], who also discuss its application to potential flow about ship hull forms. Dawson [10] developed a surface-singularity technique for the solution of appropriately linearized ship-wave problems in two and three dimensions, and Gadd [11] has considered the solution of non-linear, three-dimensional, free-surface problems by a similar technique.

The application of boundary-integral methods to linearized potential-flow problems typically results in a Fredholm integral equation of the second kind, of the form

$$\sigma(x)=H(x)+\lambda\int K(x,\xi)\sigma(\xi)\,\mathrm{d}\xi, \tag{1.1}$$

in which λ is a parameter, and the unknown function $\sigma(x)$ might represent a source strength, for example. Equation (1.1) can obviously be generalized to the case of two independent variables x and y.

Equation (1.1) is presumed to possess a unique solution $\sigma(x)$, except when the parameter λ is an eigenvalue of the homogeneous equation, obtained by setting $H(x)\equiv 0$ in equation (1.1). In this case, equation (1.1) either possesses no solution, or else the solutions cease to be unique, by the Fredholm theorems (see, for example, Courant and Hilbert [12]).

The numerical solution of equations of the form (1.1) may be carried out conveniently by the Neumann iteration scheme, which is particularly useful when limitations in computer memory preclude the direct numerical inversion of the equation. An initial estimate $\sigma^{(0)}$ is made for the unknown function, and substituted into the right-hand side of the equation. The left-hand side now yields a new estimate $\sigma^{(1)}$, which in turn may be substituted in the right-hand side to yield $\sigma^{(2)}$, and so on. It is known (e.g. Courant and Hilbert [12]) that the sequence $\{\sigma^{(n)}\}$ of approximations converges to the true solution $\sigma(x)$ provided that $|\lambda|$ is less than the absolute value of the smallest eigenvalue of the homogeneous problem. In addition, the method is often observed to converge when applied to non-linear integral equations.

In this paper, the solutions by integral-equation techniques for two-dimensional flow due to a line vortex submerged in a running stream, and for three-dimensional flow about a moving pressure distribution, are considered. In each problem, it is shown that the eigenvalues occur as a continuum lying along the entire negative real axis, rather than as discrete values. Consequently, eigenvalues of infinitesimal magnitude exist, and the Neumann iteration scheme never converges. Application of this result to the design of numerical methods for the solution of the corresponding non-linear problems is discussed.

2. Flow due to a submerged vortex

Consider a stream of infinite depth flowing from left to right with a steady speed c far upstream. A cartesian coordinate system is defined with the y-axis pointing vertically and the x-axis situated along the position of the undisturbed free surface. A vortex of strength K is present at the point $(0, -h)$, and has its circulation in the clockwise direction, following Hess [1]. The fluid is subject to the downward acceleration of gravity, g, and $\eta(x)$ denotes the position of the free surface as a function of x.

The problem is immediately non-dimensionalized by referring all lengths to the depth h and velocities to c; the velocity potential ϕ and stream-function ψ are rendered dimensionless by reference to ch. The two dimensionless parameters of the flow are the Froude number $F=c(gh)^{-1/2}$ and vortex strength $\epsilon=K(ch)^{-1}$.

The equations of linearized flow are derived, following Wehausen and Laitone ([13], p. 463), by the usual development of ϕ, ψ and η as perturbation expansions in the parameter

ϵ, of the form

$$\begin{aligned}\phi(x,y) &= x + \epsilon\phi_1(x,y) + O(\epsilon^2),\\ \psi(x,y) &= y + \epsilon\psi_1(x,y) + O(\epsilon^2),\\ \eta(x) &= \epsilon H_1(x) + O(\epsilon^2).\end{aligned} \tag{2.1}$$

Within the fluid, ϕ_1 and ψ_1 satisfy the Cauchy-Riemann equations

$$\phi_{1_x} = \psi_{1_y}, \tag{2.2a}$$

$$\phi_{1_y} = -\psi_{1_x}, \tag{2.2b}$$

except at the vortex, where the complex potential $f_1 = \phi_1 + \mathrm{i}\psi_1$ is of the form

$$f_1 \to \frac{\mathrm{i}}{2\pi}\ln(z+\mathrm{i}) \quad \text{as} \quad z \to -\mathrm{i}, \tag{2.3}$$

in terms of the complex variable $z = x + \mathrm{i}y$. The radiation condition, that no waves be present infinitely far upstream, yields the condition

$$f_1 \to 0 \quad \text{as} \quad z \to -\infty, \tag{2.4}$$

and the Bernoulli and kinematic surface conditions give rise to the well-known linearized equations

$$F^2\phi_{1_x} + H_1 = 0 \quad \text{on} \quad y = 0, \tag{2.5a}$$

and

$$H_{1_x} = \phi_{1_y} \quad \text{on} \quad y = 0. \tag{2.5b}$$

The solution of equations (2.1)-(2.5) is given by Wehausen and Laitone ([13], p. 489).

A boundary-integral formulation of the problem is now derived using the Cauchy integral theorem. (A different, but entirely equivalent, procedure was adopted by Hess [1], and consisted of distributing vorticity across the free surface). The complex potential f_1 is assumed to be of the form

$$f_1 = \frac{\mathrm{i}}{2\pi}[\ln(z+\mathrm{i}) - \ln(z-\mathrm{i})] + \mathscr{F}_1, \tag{2.6}$$

where the second term, corresponding to a vortex of opposite circulation at the image point $z = \mathrm{i}$ above the free surface, has been added to satisfy the radiation condition. Cauchy's integral formula is applied to the derivative $\chi(z) = \mathrm{d}\mathscr{F}_1/\mathrm{d}z$ of the wave-making term $\mathscr{F}_1(z)$ in equation (2.6), along the path shown in Fig. 1. Since $\chi(z)$ is continuous and vanishes as $z \to -\mathrm{i}\infty$, the contributions from either side of the branch cut cancel, and those from the semi-circular arc at infinity and the circular arc about the vortex are separately zero. In the limit as the semi-circular arc by-passing the point $x + \mathrm{i}0$ on the free surface is made to be of zero radius, the result

$$\chi(x) = -\frac{1}{\pi \mathrm{i}} ⨍_{-\infty}^{\infty} \frac{\chi(t)\mathrm{d}t}{t-x} \tag{2.7}$$

is obtained, where the improper integral is to be interpreted in the Cauchy principal-value sense.

The desired equation is obtained by taking the real part of equation (2.7), which gives

$$\phi_{1_x}(x, 0) = \frac{1}{\pi(x^2+1)} - \frac{1}{\pi} ⨍_{-\infty}^{\infty} \psi_{1_t}(t, 0) \frac{\mathrm{d}t}{t-x},$$

in view of equation (2.6). Finally, the quantity ψ_{1_x} appearing on the right-hand side may be eliminated by means of equations (2.2b) and (2.5) to yield

$$\phi_{1_x}(x, 0) = \frac{1}{\pi(x^2+1)} - \frac{F^2}{\pi} ⨍_{-\infty}^{\infty} \phi_{1_{tt}}(t, 0) \frac{\mathrm{d}t}{t-x}, \tag{2.8}$$

which is a singular, integrodifferential equation of the second kind.

The eigenvalues of equation (2.8) are constants λ satisfying the corresponding homoge-

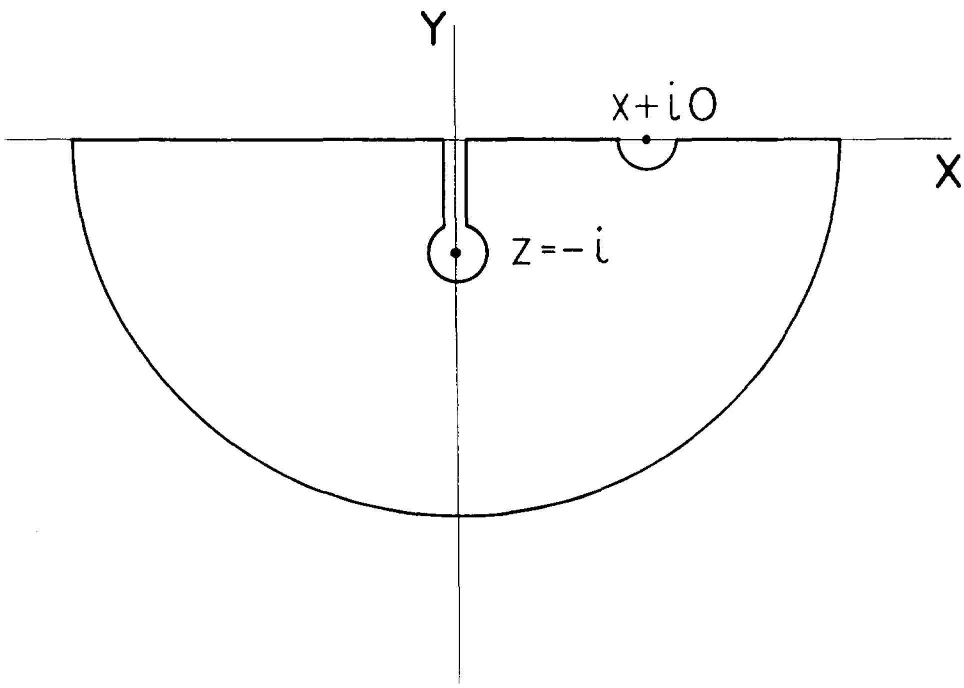

Figure 1. Contour of integration in the physical plane $z = x + \mathrm{i}y$ for the submerged vortex problem.

neous equation

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \lambda \fint_{-\infty}^{\infty} \frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} \frac{\mathrm{d}t}{t-x}, \tag{2.9}$$

where the $u(x)$ are associated eigenfunctions. Assume that

$$u(x) = \mathrm{e}^{\mathrm{i}mx}, \tag{2.10}$$

for some real $m \neq 0$; then equation (2.9) yields

$$\mathrm{i} = -m\lambda \fint_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}m\xi}\mathrm{d}\xi}{\xi}$$

for the determination of λ, after the change of variable $\xi = t - x$. The integral on the right-hand side of this expression has the value $\mathrm{i}\pi\,\mathrm{sgn}(m)$ (see Abramowitz and Stegun [14], p. 78, formula 4.3.142), and consequently,

$$\lambda = -\frac{1}{\pi|m|}. \tag{2.11}$$

Thus, *any* function of the form (2.10) is a possible eigenfunction of equation (2.8) without restriction on the values of the constant m, and by equation (2.11), any negative real number is therefore an eigenvalue. By comparison with equation (2.8), the values of the Froude number at which the Fredholm determinant vanishes are $F = |m|^{-1/2}$, and it is therefore to be expected that equation (2.8) cannot be solved by the Neumann iteration scheme for any $F \neq 0$. Notice that the frequency of a linearized wave in dimensional variables is $g(2\pi c)^{-1}$, which in terms of the present dimensionless quantities becomes $(2\pi F^2)^{-1}$; consequently, the irregular frequencies of equation (2.8) are $(2\pi)^{-1}|m|$ for any real number m.

3. Flow about a moving pressure distribution

3.1. Formulation

Consider a distribution of pressure having characteristic length $2L$ and width $2B$ moving from right to left across the surface of a fluid, with speed c relative to a stationary observer. With respect to a cartesian coordinate system moving with the pressure distribution and having the z-axis pointing vertically, the fluid flows steadily from left to right, in the direction of the positive x-axis, and its speed infinitely far upstream is c. The pressure distribution is assumed to be of the form $P_0\mathscr{P}(x, y)$, where the constant P_0 has the dimensions of pressure and represents the maximum strength of the distribution. The fluid is assumed to be inviscid and incompressible, and to flow without rotation, and is subject to the downward acceleration of gravity, g.

Dimensionless variables are defined by choosing c as a reference velocity and L as a reference length. The velocity potential ϕ is made dimensionless with respect to the product cL. A particular flow is thus characterized by the length-based Froude number

$$F = \frac{c}{(gL)^{1/2}},$$

the dimensionless half-width

$$\beta = \frac{B}{L}$$

and the strength

$$\alpha = \frac{P_0}{\rho g L}$$

of the pressure distribution.

In the interior of the fluid, the velocity potential ϕ satisfies Laplace's equation

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \tag{3.1}$$

and is also subject to the condition that the flow be uniform at infinite depth within the fluid. If the unknown free-surface elevation is described by the equation $z = \zeta(x, y)$, then the usual kinematic free-surface condition and Bernoulli equation may be written

$$\phi_x\zeta_x + \phi_y\zeta_y = \phi_z \quad \text{on} \quad z = \zeta \tag{3.2}$$

and

$$\tfrac{1}{2}F^2\left(\phi_x^2 + \phi_y^2 + \phi_z^2\right) + \zeta + \alpha\mathscr{P} = \tfrac{1}{2}F^2 \quad \text{on} \quad z = \zeta, \tag{3.3}$$

respectively. The pressure distribution function $\mathscr{P}(x, y)$ is assumed to vanish as $x^2 + y^2 \to \infty$, and so the radiation condition, that no waves be present far upstream, can be written

$$\phi_x \to 1,\ \phi_y \to 0,\ \phi_z \to 0 \quad \text{and} \quad \zeta \to 0 \quad \text{as} \quad x \to -\infty. \tag{3.4}$$

The dependent variables ϕ and ζ are now expressed as regular perturbation expansions in the parameter α, analogously to equations (2.1). For completeness, we shall retain the higher-order terms, giving

$$\begin{aligned} \phi(x, y, z) &= x + \sum_{j=1}^{\infty} \alpha^j \Phi_j(x, y, z), \\ \zeta(x, y) &= \sum_{j=1}^{\infty} \alpha^j Z_j(x, y). \end{aligned} \tag{3.5}$$

Clearly each function Φ_j satisfies Laplace's equation (3.1), subject to the conditions

$$\Phi_j \to 0,\ \nabla\Phi_j \to \mathbf{0} \quad \text{as} \quad z \to -\infty, \tag{3.6}$$

and the radiation condition (3.4) gives

$$\nabla \Phi_j \to \mathbf{0} \quad \text{and} \quad Z_j \to 0 \quad \text{as} \quad x \to -\infty. \tag{3.7}$$

The functions Φ_j evaluated at the free surface $z = \zeta(x, y)$ are expressed in terms of Taylor-series expansions about the plane $z = 0$, as in Section 2. The kinematic condition (3.2) may thus be written symbolically as

$$Z_{1_x}(x, y) = \Phi_{1_z}(x, y, 0), \tag{3.8a}$$

$$Z_{j_x}(x, y) + R_j(x, y) = \Phi_{j_z}(x, y, 0), j \geq 2, \tag{3.8b}$$

and the Bernoulli equation (3.3) yields

$$F^2 \Phi_{1_x}(x, y, 0) + Z_1(x, y) + \mathscr{P}(x, y) = 0, \tag{3.9a}$$

$$F^2 \Phi_{j_x}(x, y, 0) + Z_j(x, y) + Q_j(x, y) = 0, j \geq 2, \tag{3.9b}$$

where the quantities R_j and Q_j are complicated functions of sums of products of lower-order terms and their derivatives.

The integral equations satisfied at each order of the expansion (3.5) are derived by applying Green's second formula to the functions Φ_j, within the volume V shown in Fig. 2. This volume is contained within the hemi-spherical surface S_∞ at infinite depth within the fluid, the surface S_F which consists simply of the plane $z = 0$ punctured by a small disk of radius ϵ centred at the point Q, and the hemi-spherical surface S_ϵ of radius ϵ and centre

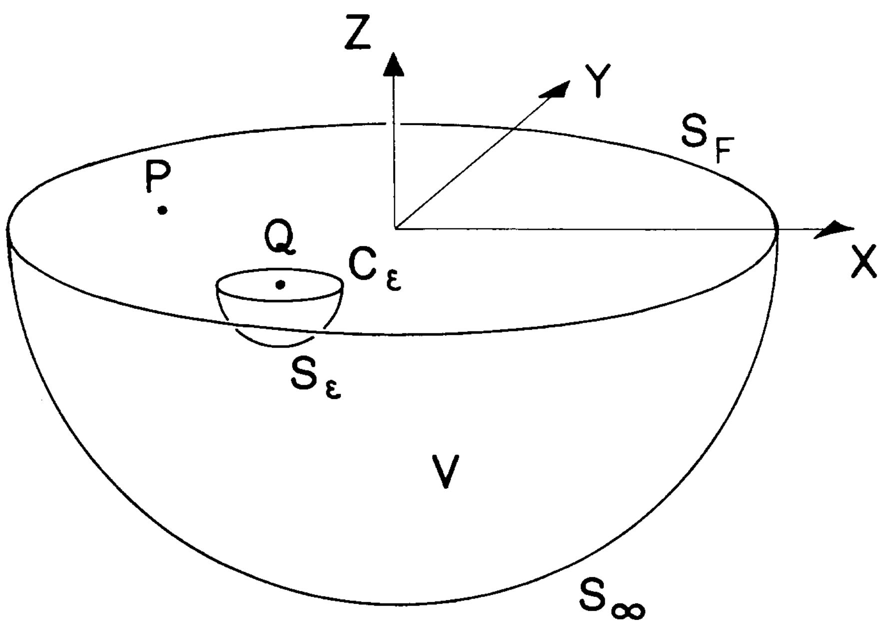

Figure 2. Domain of integration used to derive the integral equations for the pressure distribution problem.

Q. The singular solution of Laplace's equation (3.1) is taken to be a Green function of the form

$$G(P, Q) = \frac{1}{R_{PQ}} + \mathscr{H}(P, Q), \tag{3.10}$$

where

$$R_{PQ} = \left[(\rho - x)^2 + (\sigma - y)^2 + (\tau - z)^2\right]^{1/2}$$

is the distance between points $P(\rho, \sigma, \tau)$ and $Q(x, y, z)$, and $\mathscr{H}$ is a harmonic function which is regular in the volume V and on its surface. Thus

$$\oiint_S \left(\Phi_j \frac{\partial G}{\partial n} - G \frac{\partial \Phi_j}{\partial n} \right) \mathrm{d}S = 0, \tag{3.11}$$

where $S = S_\infty + S_F + S_\epsilon$ is the surface of volume V, and $\mathbf{n}$ is a unit normal vector to S, chosen to point into volume V.

Equation (3.6) indicates that the portion of the integral in equation (3.11) over the surface S_∞ vanishes, leaving only the contributions from surfaces S_F and S_ϵ. Since $\mathscr{H}(P, Q)$ is continuous as $P \to Q$, the contribution from surface S_ϵ becomes

$$\lim_{\epsilon \to 0} \iint_{S_\epsilon} \left(-\frac{\Phi_j}{\epsilon^2} - \frac{1}{\epsilon} \frac{\partial \Phi_j}{\partial n} \right) \mathrm{d}S,$$

which has the limiting value $-2\pi\Phi_j(Q)$. On the surface S_F, $\partial/\partial n = -\partial/\partial z$, so that equation (3.11) becomes

$$2\pi\Phi_j(Q) = -\iint_{S_F} \left\{ \Phi_j(P) \frac{\partial G(P, Q)}{\partial \tau} - G(P, Q) \frac{\partial \Phi_j(P)}{\partial \tau} \right\} \mathrm{d}\rho\, \mathrm{d}\sigma. \tag{3.12}$$

The integral equation for the linearized problem is derived from equation (3.12) with the use of (3.8a) and (3.9a) and becomes

$$2\pi\Phi_1(Q) = -\iint_{S_F} G(P, Q)\mathscr{P}_\rho(P)\mathrm{d}\rho\,\mathrm{d}\sigma$$

$$- \iint_{S_F} \left\{ \Phi_1(P) \frac{\partial G(P, Q)}{\partial \tau} + F^2 G(P, Q) \frac{\partial^2 \Phi_1(P)}{\partial \rho^2} \right\} \mathrm{d}\rho\, \mathrm{d}\sigma. \tag{3.13a}$$

Similarly, equations (3.8b) and (3.9b) yield the integral equations

$$2\pi\Phi_j(Q) = \iint_{S_F} G(P, Q)\left[R_j(P) - Q_{j_\rho}(P)\right]\mathrm{d}\rho\,\mathrm{d}\sigma$$

$$- \iint_{S_F} \left\{ \Phi_j(P) \frac{\partial G(P, Q)}{\partial \tau} + F^2 G(P, Q) \frac{\partial^2 \Phi_j(P)}{\partial \rho^2} \right\} \mathrm{d}\rho\, \mathrm{d}\sigma \tag{3.13b}$$

for the higher-order functions Φ_j, $j \geq 2$.

3.2. Simple (Rankine) source function

The Rankine source potential is defined by setting $\mathscr{H}(P, Q) = 0$ in equation (3.10). Now require that both $P(\rho, \sigma, 0)$ and $Q(x, y, 0)$ be restricted to the surface S_F, and define

$$H_1(Q) = -\frac{1}{2\pi}\iint_{S_F}\frac{\mathscr{P}_\rho(P)}{R_{PQ}}\mathrm{d}\rho\mathrm{d}\sigma$$

and

$$H_j(Q) = \frac{1}{2\pi}\iint_{S_F}\frac{1}{R_{PQ}}\left[R_j(P) - Q_{j_\rho}(P)\right]\mathrm{d}\rho\mathrm{d}\sigma$$

for $j \geq 2$; then (3.13) yield the integrodifferential equations

$$\Phi_j(Q) = H_j(Q) - \frac{F^2}{2\pi}\iint_{S_F}\frac{1}{R_{PQ}}\frac{\partial^2\Phi_j(P)}{\partial\rho^2}\mathrm{d}\rho\mathrm{d}\sigma \tag{3.14}$$

for each of the functions Φ_j, $j \geq 1$.

The eigenvalues of equations (3.14) are determined from the homogeneous equation

$$u(x, y) = \lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u_{\rho\rho}(\rho, \sigma)}{\left[(\rho - x)^2 + (\sigma - y)^2\right]^{1/2}}\mathrm{d}\rho\mathrm{d}\sigma, \tag{3.15}$$

where λ and $u(x, y)$ are the eigenvalue and eigenfunction, respectively, and the punctured surface S_F has been replaced by the complete plane $z = 0$, in view of the weak (integrable) nature of the singularity of the kernel R_{PQ}^{-1}. Assume that

$$u(x, y) = \mathrm{e}^{\mathrm{i}mx + \mathrm{i}ny}, \tag{3.16}$$

for real numbers $m, n \neq 0$; then, after the changes of variable $s = \rho - x$, $t = \sigma - y$, equation (3.15) becomes

$$1 = -\lambda m^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}ms + \mathrm{i}nt}}{[s^2 + t^2]^{1/2}}\mathrm{d}s\mathrm{d}t.$$

The integral on the right-hand side of this equation has the known value $2\pi(m^2 + n^2)^{-1/2}$, which may be shown by multiplying the integrand by $\exp(-c[s^2 + t^2]^{1/2})$, $c > 0$, changing to polar variables, then evaluating the integral by the calculus of residues in the limit $c \to 0$. Consequently,

$$\lambda = -\frac{(m^2 + n^2)^{1/2}}{2\pi m^2}, \tag{3.17}$$

which is a natural generalization of equation (2.11) for three-dimensional geometry. Evidently *any* function of the form (3.16) is an admissible eigenfunction of equation

(3.15), and equation (3.17) again indicates that the eigenvalues form a continuum on the entire negative real axis.

3.3. Havelock Green function

We now investigate the consequences of utilizing the classical source function of Havelock as the singular solution of Laplace's equation. This is a function $G(P, Q)$ which, in addition to satisfying Laplace's equation (3.1) and boundary conditions (3.6), also satisfies the linearized surface condition

$$\frac{\partial G(P,Q)}{\partial \tau} + F^2 \frac{\partial^2 G(P,Q)}{\partial \rho^2} = 0. \tag{3.18}$$

In order to remain consistent with equation (3.10), the function $G(P, Q)$ meeting the above requirements is written in the form

$$G(P,Q) = \frac{1}{R_{PQ}} - \frac{1}{R'_{PQ}} - \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\exp\left(k\{(z+\tau) + \mathrm{i}[(x-\rho)\cos\theta + (y-\sigma)\sin\theta]\}\right)}{kF^2\cos^2\theta - 1}\,\mathrm{d}k\,\mathrm{d}\theta, \tag{3.19}$$

where

$$R'_{PQ} = \left[(\rho-x)^2 + (\sigma-y)^2 + (\tau+z)^2\right]^{1/2}.$$

The function $G(P, Q)$ in equation (3.19) is the well-known Green function of Havelock; although the integral on the right-hand side is formally divergent, due to the presence of a pole singularity in the integrand, the expression is rendered meaningful by interpreting the improper integral (with respect to k) in the Cauchy principal-value sense, and then adding to it some multiple of a free-wave term, chosen to satisfy the radiation condition (3.7). The result is classical, and is given by Wehausen and Laitone ([13], p. 484). In the following, it will be assumed that such an interpretation has already been given to equation (3.19).

The double-integral term in equation (3.19) is not well-behaved when $\tau \to 0$ for $z = 0$, and consequently, equation (3.18) is *not* satisfied in this limit. Curiously, the correct surface condition satisfied by this Green function when $z = 0$ appears to have been given in explicit form only quite recently. (See Noblesse [15]). The integral equations (3.13) are not satisfied for this choice of Green function since $\mathscr{H}(P, Q)$ is no longer continuous as $P \to Q$ when $z = 0$, and to derive the correct result it is necessary to return to equation (3.11).

By the boundary condition (3.6), the contribution from surface S_∞ is again zero. However, the integral over surface S_ϵ is now also zero, as may be seen by transforming to spherical polar coordinates and then allowing $\epsilon \to 0$. Thus equation (3.11) yields simply

$$\iint_{S_F}\left(\Phi_j\frac{\partial G}{\partial \tau} - G\frac{\partial \Phi_j}{\partial \tau}\right)\mathrm{d}\rho\mathrm{d}\sigma = 0,$$

which, in view of the surface conditions (3.8) and (3.9) satisfied by the functions Φ_j and the condition (3.18) satisfied by $G(P, Q)$ for $P \neq Q$, gives

$$F^2\iint_{S_F}\left(\Phi_1\frac{\partial^2 G}{\partial \rho^2} - G\frac{\partial^2 \Phi_1}{\partial \rho^2}\right)\mathrm{d}\rho\mathrm{d}\sigma = \iint_{S_F} G\frac{\partial \mathscr{P}}{\partial \rho}\mathrm{d}\rho\mathrm{d}\sigma \tag{3.20a}$$

and

$$F^2\iint_{S_F}\left(\Phi_j\frac{\partial^2 G}{\partial \rho^2} - G\frac{\partial^2 \Phi_j}{\partial \rho^2}\right)\mathrm{d}\rho\mathrm{d}\sigma = \iint_{S_F} G\left(\frac{\partial Q_j}{\partial \rho} - R_j\right)\mathrm{d}\rho\mathrm{d}\sigma \quad \text{for } j \geq 2. \tag{3.20b}$$

Following Miloh and Landweber [16], the integrals on the left-hand sides of equations (3.20) are transformed according to the identity

$$\iint_{S_F}\left(\Phi_j\frac{\partial^2 G}{\partial \rho^2} - G\frac{\partial^2 \Phi_j}{\partial \rho^2}\right)\mathrm{d}\rho\mathrm{d}\sigma = -\oint_{C_\epsilon}\left(\Phi_j\frac{\partial G}{\partial \rho} - G\frac{\partial \Phi_j}{\partial \rho}\right)l_\epsilon \mathrm{d}s, j \geq 1, \tag{3.21}$$

which follows after integration by parts and the application of Green's theorem in the plane. The circular path C_ϵ formed by the intersection of surfaces S_ϵ and S_F is traversed in the anti-clockwise direction, and l_ϵ and $\mathrm{d}s$ denote the direction cosine of the normal to C_ϵ with the x-axis, and an element of arc-length on C_ϵ, respectively.

The integral on the right-hand side of equation (3.21) is evaluated in the limit $\epsilon \to 0$ by parametrizing the curve C_ϵ and observing that, since $\partial\Phi_j/\partial x$ is continuous at the point Q, the second term vanishes, leaving

$$\oint_{C_\epsilon}\left(\Phi_j\frac{\partial G}{\partial \rho} - G\frac{\partial \Phi_j}{\partial \rho}\right)l_\epsilon \mathrm{d}s$$

$$\to \lim_{\epsilon \to 0}\frac{\mathrm{i}\epsilon}{\pi}\Phi_j(Q)\int_0^{2\pi}\cos t\mathrm{d}t\int_0^{2\pi}\int_0^{\infty}\frac{k\cos\theta e^{-\mathrm{i}k\epsilon\cos(t-\theta)}}{kF^2\cos^2\theta - 1}\mathrm{d}k\mathrm{d}\theta. \tag{3.22}$$

The change of variable $\xi = k\epsilon$ enables the explicit evaluation of the limit $\epsilon \to 0$, giving

$$\frac{\mathrm{i}}{\pi F^2}\Phi_j(Q)\int_0^{2\pi}\cos t\mathrm{d}t\int_0^{2\pi}\int_0^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\xi\cos(t-\theta)}}{\cos\theta}\mathrm{d}\xi\mathrm{d}\theta.$$

The improper integral with respect to ξ may be evaluated immediately by multiplying the integrand by $\mathrm{e}^{-c\xi}$, $c > 0$, and the integrations with respect to t and θ are then interchanged. Since the limit $c \to 0$ is ultimately intended, the above expression becomes

$$\lim_{c \to 0}\frac{\mathrm{i}\Phi_j(Q)}{\pi F^2}\int_0^{2\pi}\frac{\mathrm{d}\theta}{\cos\theta}\int_0^{2\pi}\frac{\cos t\mathrm{d}t}{c + \mathrm{i}(\cos t\cos\theta + \sin t\sin\theta)}.$$

The innermost integration may be shown to give the result $(2\pi \cos\theta)/\mathrm{i}$ for $c \to 0$, using the calculus of residues, so that equation (3.22) finally yields

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon}\left(\Phi_j\frac{\partial G}{\partial\rho} - G\frac{\partial\Phi_j}{\partial\rho}\right) l_\epsilon \mathrm{d}s = \frac{4\pi\Phi_j(Q)}{F^2}. \tag{3.23}$$

Equations (3.21) and (3.23) are substituted into the left-hand sides of equations (3.20). The term on the right-hand side of equation (3.20a) is integrated by parts, and the final expressions are

$$\Phi_1(Q) = \frac{1}{4\pi}\iint_{S_F}\mathscr{P}(\rho,\sigma)G_\rho(\rho,\sigma,0,x,y,0)\mathrm{d}\rho\mathrm{d}\sigma \tag{3.24a}$$

and

$$\Phi_j(Q) = \frac{1}{4\pi}\iint_{S_F}\left[R_j(\rho,\sigma) - Q_{j_\rho}(\rho,\sigma)\right]G(\rho,\sigma,0,x,y,0)\mathrm{d}\rho\mathrm{d}\sigma, j\geq 2. \tag{3.24b}$$

Thus the choice of the Havelock Green function (3.19) as the singular solution to Laplace's equation gives rise to explicit expressions for the functions Φ_j at each order, rather than integral equations. The expression (3.24a) is the well-known linearized solution given by Wehausen and Laitone ([13], p. 598).

4. Discussion and conclusions.

Two-dimensional flow due to a vortex submerged in a running stream and three-dimensional flow induced by a moving distribution of pressure on the surface of a fluid have been investigated. In each case, the linearized problem formulated in terms of the fundamental singular solution to Laplace's equation yields a linear, singular, integrodifferential equation for the velocity potential, the eigenvalues of which occur as a continuum encompassing the entire negative real axis. In a general setting, this result is entirely to be expected both from the physical and mathematical viewpoints; physically, it is simply a statement that standing waves of *any* frequency or wavelength are possible in an unbounded fluid, whilst mathematically, it is well known that non-compact (integral) operators may have a continuous spectrum. However, although the result is scarcely surprising, it is of considerable importance in the attempted numerical solution of these problems, which is the principal interest of the present study. In particular, it is clear that the Neumann iteration scheme, or some numerical equivalent of it, can *never* converge, regardless of the initial guess.

To the extent that closed-form linearized solutions are known to both of the problems formulated in this study, the numerical solution of the linearized equations is of little interest. However, difficulties similar to those encountered in the attempted solution of the linearized equations persist also in the non-linear case, and we are prepared to offer the following conjecture concerning the properties of the corresponding non-linear integral equation for ϕ on the exact (unknown) free-surface location:

CONJECTURE: Suppose the series (3.5) (or (2.1)) and the Taylor-series expansion of the velocity potential at the exact free surface in terms of quantities at the undisturbed surface level are both uniformly convergent whenever $|\alpha| < r_c$. Then, for each eigenvalue of the linearized equation, there corresponds a unique element in the spectrum of the non-linear equation, at least when $|\alpha| < r_c$.

The basis for this conjecture is the observation that the higher-order equations (3.8b) and (3.9b) in the expansion of the kinematic and Bernoulli conditions involve the identical differential operators to the linearized equations (3.8a) and (3.9a), and consequently, each integrodifferential equation (3.13b) involves the same integral operator as the linearized equation (3.13a). (This may also be shown to be true for the vortex problem in Section 2.) Three immediate consequences of this conjecture are now examined.

Firstly, the above conjecture indicates that, if the Green function is chosen to be the fundamental solution to Laplace's equation, then the *non-linear* integrodifferential equation should possess an infinite continuum of eigenvalues, similar to the behaviour of the linearized equations investigated in Sections 2 and 3. This would likewise be manifested in the failure of the Neumann scheme to converge to a solution at any Froude number, and for any initial guess. In order to examine the accuracy of this prediction, the non-linear equations for the case of flow due to a moving pressure distribution were programmed on a PRIME mini-computer, and their solution was attempted using the Neumann iteration process, for a wide variety of Froude numbers, and employing various different initial guesses. The iterates were ultimately observed to diverge in every case, as expected.

A second consequence of this conjecture concerns instances in which the linearized integrodifferential equation possesses at most countably infinitely many eigenvalues, rather than a continuous distribution of them. This might be achieved either by the selection of a different kernel function, or perhaps by an appropriate truncation of the domain of integration. Then, for $|\alpha| < r_c$, the non-linear equation is also expected to possess a countable spectrum, with each element corresponding to a unique eigenvalue of the linearized problem.

The convergence difficulties encountered by Hess [1] may now be analyzed in the light of the foregoing remarks. Hess sought to solve the non-linear problem of flow due to a submerged line vortex using what appears to be a variant of the method of Hess and Smith [7,8]. This method employs an iteration procedure which is numerically equivalent to the Neumann scheme, and consequently, may be expected always to diverge, in view of the above conjecture and the fact that the linearized integral operator was shown in Section 2 to possess a continuous spectrum. This divergence was noted by Hess, but was nevertheless circumvented by the additional imposition of an "initial flat" on the problem, in which a portion of the free surface upstream of the vortex was arbitrarily chosen to be flat. Evidently this requirement is equivalent to a truncation of the domain of integration of the non-linear integral operator, which, by the above remarks, presumably now possesses a countable spectrum. Consequently, the method of Hess and Smith converges in a radius determined by the smallest eigenvalue of the non-linear operator.

In a recent paper, Ursell [17] has shown how irregular frequencies in a Fredholm integral equation of the second kind may be eliminated by a judicious choice of the Green function (3.10). The third consequence of the above conjecture pertains to this result, since then it follows that, at least for $|\alpha| < r_c$, such a choice of Green function evidentally eliminates irregular frequencies from the non-linear problem as well. Indeed, for the simple problems investigated in this study, the choice of the Havelock Green function

(3.19) (or the corresponding two-dimensional form given by Ursell [17]) leads to explicit solutions (3.24) at each order of the expansion (3.5), rather than integral equations. Equations (3.24) could be evaluated numerically and the series (3.5) carried in principle to any desired order, although in practice the difficulty of performing the numerical quadratures with accuracy, as well as the complexity of the functions $R_j(x, y)$ and $Q_j(x, y)$, $j \geq 2$, would appear to disqualify this semi-numerical approach as a viable method for the solution of the fully non-linear problem. Nevertheless, the evaluation of the second- and third-order terms might be valuable, and an approach similar to this is advocated by Miloh and Landweber [16].

Since the choice of the Havelock Green function evidently eliminates irregular frequencies from both the linearized and non-linear problems, the latter could be formulated as an exact integral equation on the unknown free surface, with the Havelock function as its kernel. This non-linear equation should then yield to numerical solution by the Neumann iteration scheme. However, it is by no means clear whether the considerable advantage afforded by the use of this iteration scheme would adequately compensate for the enormous computational effort of evaluating the Green function (3.19) and its derivatives at each new iteration. In addition, this formulation might lead to ill-conditioning, in the sense that the Havelock function would need to be known to high accuracy to avoid divergence of the iterates.

An alternative approach, based on the numerical method of Forbes and Schwartz [4], has been developed for the solution of the fully non-linear problem described in Section 3, and appears to give good results. An integral equation for the velocity potential is formulated in terms of the fundamental singularity R_{PQ}^{-1}, as in Section 3.2, and is solved by a Newtonian iteration process. Details of this method will be presented in a future publication.

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